Timed symbolic dynamics*

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Abstract. We introduce a theory of timed symbolic dynamics unifying results from timed automata theory and symbolic dynamics. The timed sofic shift spaces we define are a way of seeing timed regular languages as shift spaces on general alphabets (in classical symbolic dynamics, sofic shift spaces correspond to regular languages). We show that morphisms of shift spaces on general alphabets can be approximated by sliding block codes resulting in a generalised version of the so-called Curtis-Hedlund-Lyndon Theorem. We provide a new measure for timed languages by characterising the Gromov-Lindenstrauss-Weiss metric mean dimension for timed shift spaces and illustrate it on several examples. We revisit recent results on volumetry of timed languages in terms of timed symbolic dynamics. In particular we explain the discretisation of timed shift spaces and their entropy.

1 Introduction

Timed automata were introduced in the early 1990's to model continuous time behaviours in a verification context [1]. Since then they have been thoroughly studied from a theoretical standpoint, a common challenge being the lifting of results from the well established automata theory.

The theory of symbolic dynamics was developed since the beginning of the 20th century as a method to study, in a symbolic fashion, general dynamical systems like ordinary differential equations. The method consists in associating an (infinite) sequence of symbols to every (infinite) trajectory of the dynamical system. The symbols represents regions of a finite partition of the state space that are visited along trajectories at discrete time steps. Departing from its topological origins, symbolic dynamics has a lot of applications in channel-coding theory and data storage, number theory, and linear algebra (see [19] and reference therein). It is also used in the context of analysis of algorithm (see e.g. [26]).

Thus, as automata theory, symbolic dynamics is a broad research field that can provide a source of interesting results to lift to the timed world. Indeed, this

* This research is supported in part by ERC Advanced Grant VERIWARE and was also supported by the ANR project EQINOCS (ANR-11-BS02-004). The present article is an improved and shortened version of Chapter 5 of the PhD thesis [10]. Omitted proofs and extra details can be found in the appendix.

theory has three interesting characteristics. (i) It is really close to automata theory and dealing with object very similar to regular languages (namely the set of allowed block of sofic shifts) and having similar results such as determinisation, minimisation, pumping lemma, etc. (ii) Symbolic dynamics provides a quantitative analysis of regular languages with the notion of entropy. Entropy measures the growth rate of the languages with respect to the size of words considered. (iii) Symbolic dynamics considers regular languages as dynamical systems called shift spaces and provides a topological point of view. For instance the entropy of a shift space is a particular case of the so-called topological entropy defined for general dynamical systems (see e.g. [15]). Thus this theory is nicely placed within broad mathematical theories developed by the pioneers Markov, Shannon, Kolmogorov.

Volumes and volumetric entropy were recently introduced [6,4] to quantify the size of timed languages and the information content of their elements. These exploratory works were inspired by the symbolic dynamics' notion of entropy but left open several questions. What is the notion of shift space for timed automata? How is the topological entropy of such an hypothetical shift space linked to the volumetric entropy? What are the others quantitative results that can be borrowed from symbolic dynamics?

Contributions. Here, we propose a theory of timed symbolic dynamics that sheds a new light on the underlying dynamics of timed regular languages (the languages recognised by timed automata). The main difficulty here is that the natural shift space for a timed language has an infinite, and even uncountable, alphabet. Such shift spaces are quite different from those usually studied in symbolic dynamics. Thus we first define and characterise shift spaces on general alphabets that are compact, metric and measurable spaces. Then we associate general alphabet shift spaces to timed languages and study their properties; we call timed sofic shift such shift spaces. As for size/complexity measures for timed sofic shift, the standard approach based on topological entropy cannot work: this entropy is infinite, and we study more relevant characteristics. The first one is the Gromov-Lindenstrauss-Weiss metric mean dimension [20] that we characterise for timed sofic shift. The second one is obtained by "renormalisation" of topological entropy, and turns out to coincide with volumetric entropy of timed languages (thus we justify in terms of symbolic dynamics the somewhat ad hoc definitions from [6,4]). We also investigate morphisms for general alphabet shift spaces, namely, we state a generalisation of the Curtis-Hedlund-Lyndon theorem after proving that the classical statement cannot hold for general alphabet shift spaces.

Related work and possible applications. This article is designed for readers with a basic knowledge of automata theory or of symbolic dynamics. No specific knowledge of these fields is required to read the paper. We refer the reader to [19] for an extensive introduction to symbolic dynamics and to [13] for an exposition of this theory in the context of automata theory. The seminal paper on timed automata is [1].

In [27], deterministic continuous dynamical systems are abstracted by timed automata. Hence the state space is discretised as in symbolic dynamics while the timing behaviours are made non-deterministic because of the abstraction. It would be very interesting to use symbolic dynamics methods in this context.

We introduced the basis of a timed theory of channel coding in [2]. Algorithms and result of this latter paper were inspired by the theory of symbolic dynamics and coding. We think that the formal exposition of timed symbolic dynamics in the present paper will probably lead to new developments of the timed theory of channel coding.

Metric mean dimension can be interpreted in terms of robustness analysis as follows: it tells us how often arbitrary precision is required to encode delays along timed words. For instance a timed automaton with metric mean dimension $1/3$ means that $2/3$ of the delays must be chosen with arbitrary precision. A more detailed discussion with related work is given in the conclusion (Section 5).

Article structure. After giving preliminaries in Section 2, we characterise properties of general alphabet shift spaces in Section 3. In particular, we explain why several key results of symbolic dynamics do not hold in this more general settings (Fact 1 and 2) and what are the suitable generalisations of these results (definition of the volumetric entropy and Theorem 3). In Section 4, we associate general alphabet shift spaces to timed automata and study there quantitative properties (entropy, metric mean dimension). In Section 5 we discuss the results obtained in this paper and the perspectives.

2 Preliminaries

In this section we give topological definitions of shift spaces from symbolic dynamics (see [19,15]) except that we generalise definitions from finite to compact metric alphabets in Section 2.3. We use classical topology concepts whose definitions and properties can be found in books such as [21].

2.1 Dynamical systems

Let (X, d) be a compact metric space. A subset $Y \subseteq X$ is an *ε-net* of X if every element of X is at most ε far apart from an element of Y ($\forall x \in X,\ \exists y \in Y$ such that $d(x, y) \leq \varepsilon$). In a compact set, for all $\varepsilon > 0$, there exists a finite ε-net of it. We denote by $\mathcal{N}_\varepsilon(X, d)$ the minimal cardinality of ε-net of X. A subset $Y \subseteq X$ is *ε-separated* if all two different elements of Y are at least ε far apart from each other ($\forall x, y \in Y,\ x \neq y \Rightarrow d(x, y) > \varepsilon$). In a compact set, all ε-separated set are of finite cardinality. We denote by $\mathcal{S}_\varepsilon(X, d)$ the maximal cardinality of ε-separated sets of X.

Lemma 1 ([18], see also [15] for an English version). *Given a compact metric space X the followings inequalities hold $\mathcal{S}_{2\varepsilon}(X, d) \leq \mathcal{N}_\varepsilon(X, d) \leq \mathcal{S}_\varepsilon(X, d)$.*

A *discrete time dynamical system* (just called dynamical system thereafter) is a couple $((X,d),f)$ where (X,d) is a compact metric space and f is a homeomorphism of X i.e. a continuous bijection from X to X. Informally, we can see X as the state space of the system. The function f is the evolution law of the system, it gives the dynamics: given a starting state x_0, the states $f(x_0), f^2(x_0), \ldots$ are the successors of x, $f^n(x_0)$ is the state at the "moment" n. The function f^{-1} permits one to go back in the past. A continuous function ϕ from a dynamical system $((X,d),f)$ to another $((X',d'),f')$ that commutes with the dynamics (i.e. $f' \circ \phi = \phi \circ f$) is called a *morphism*.

2.2 ε-entropies and topological entropy

The topological entropy permits one to measure the complexity of a system. Intuitively a system is complex when it is sensitive to initial conditions. There are several equivalent ways to define topological entropy, here we give a definition due to Bowen [14]. Let $((X,d),f)$ be a dynamical system. For all positive integer n we define the distance between the n first iterations of f on $x,y \in X$ by:

$$d_n(x,y) = \max_{0 \leq k \leq n-1} d(f^k(x), f^k(y)).$$

The idea is that two points x and y are ε far apart for d_n if when iterating f at most n times, we can distinguish them with a precision ε. An ε-net[1] for d_n is thus an approximation of the system during n iterations and with precision ε. The N-ε entropy $h_\varepsilon^N(X)$ measures the growth rate of these sets wrt. n:

$$h_\varepsilon^N(X) = \limsup_{n \to \infty} \frac{1}{n} \log_2(\mathcal{N}_\varepsilon(X, d_n)).$$

Similarly the S-ε-entropy is:

$$h_\varepsilon^S(X) = \limsup_{n \to \infty} \frac{1}{n} \log_2(\mathcal{S}_\varepsilon(X, d_n)).$$

The topological entropy is:

$$h_{\text{top}}(X) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} h_\varepsilon^N(X) = \lim_{\varepsilon \to 0} h_\varepsilon^S(X). \tag{1}$$

The second equality is due to Lemma 1. A real ε is called a *discretisation step* if it is the inverse of a positive integer. In the following we consider wlog. only reals ε that are discretisation steps (they provide sequences that tend to 0).

2.3 Shift spaces on general alphabet

In the broad field of research of symbolic dynamics (see [19]), the shift spaces considered are on finite alphabets (we give an example in Section 2.4 below).

[1] (X, d_n) is a metric compact space when (X,d) is; so, one can consider ε-net of it.

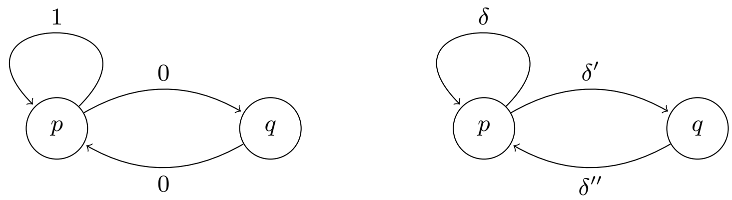

Fig. 1. A labelled graph (left) and its unlabelled version (right)

Here, we present shift spaces in a version extended to general alphabets being compact metric spaces. The main instantiation in the following is the *timed alphabet* $[0, M] \times \Sigma$ where $M \in \mathbb{N}$ and Σ is finite, with metric $d((t, a), (t', a')) = |t - t'| + 1_{a \neq a'}$. In the rest of the article $(\mathcal{C}, d)$ is a compact metric alphabet. We denote by $\mathcal{C}^{\mathbb{Z}}$ the set of bi-infinite words over $\mathcal{C}$ (i.e. words of the form $x = (x_i)_{i \in \mathbb{Z}}$ with $x_i \in \mathcal{C}$). One can define a metric $\overline{d}$ on $\mathcal{C}^{\mathbb{Z}}$ by $\overline{d}(x, x') = \sup_{i \in \mathbb{Z}} \frac{d(x_i, x'_i)}{2^{|i|}}$. The *shift map* σ is defined by $y = \sigma(x)$ when for all $i \in \mathbb{Z},\ y_i = x_{i+1}$.

It can easily be shown that $((\mathcal{C}^{\mathbb{Z}}, \overline{d}), \sigma)$ is a dynamical system, we call it the *full shift space* on $\mathcal{C}$ and just denote it by $\mathcal{C}^{\mathbb{Z}}$ when d and σ are clear from the context. A subspace X of $\mathcal{C}^{\mathbb{Z}}$ is called a *sub-shift space* of $\mathcal{C}^{\mathbb{Z}}$ whenever it is topologically closed and shift invariant: $\sigma(X) = X$. We often just call shifts (or shift spaces) the sub-shift spaces of full shift spaces.

Given a bi-infinite word $x \in \mathcal{C}^{\mathbb{Z}}$ and two indices $i, j \in \mathbb{Z}$ with $i \leq j$, the finite word $x_i x_{i+1} \cdots x_j$ is called a *factor* of x and is denoted by $x_{[i..j]}$. For a shift space X, the set of factors of length n of bi-infinite words of X is denoted by $X_n \stackrel{\text{def}}{=} \{x_{[i+1..i+n]} \mid x \in X,\ i \in \mathbb{Z}\}$.

2.4 Edge and sofic shifts from classical symbolic dynamics

Here, we recall the definitions of edge and sofic shift central in symbolic dynamics. These definitions will be lifted to the timed setting in Section 4.

Let $G = (Q, \Delta)$ be a finite graph with possibly multiple edges between two vertices. Any edge $\delta \in \Delta$ has an *origin* $\delta^- \in Q$ and a *destination* $\delta^+ \in Q$. Let Σ be a finite alphabet and $\mathtt{Lab} : \Delta \to \Sigma$ a labelling function on edges. The pair $(G, \mathtt{Lab})$ is called a *labelled graph*.

A finite (resp bi-infinite) path of G is a finite (resp bi-infinite) sequence of consecutive edges δ_i such that for all $i \in \{1, \dots, n-1\}$ (resp $i \in \mathbb{Z}$) ${\delta_i}^+ = {\delta_{i+1}}^-$.

The set of bi-infinite paths of a graph G is a sub-shift of $\Delta^{\mathbb{Z}}$ called the *edge shift* of G. The *sofic shift* of a labelled graph $A = (G, \mathtt{Lab})$ is the set of bi-infinite words that label bi-infinite paths of A: $[A] \stackrel{\text{def}}{=} \{(\mathtt{Lab}(\delta_i))_{i \in \mathbb{Z}} \mid \forall i \in \mathbb{Z}, {\delta_i}^+ = {\delta_{i+1}}^-\}$. It is a sub-shift of $\Sigma^{\mathbb{Z}}$. A labelled graph is called *right-resolving* whenever for every vertex q, all edges starting from q have distinct labels.

Example 1. Consider the graph G on the right of figure 1 and $\mathtt{Lab}$ the labelling function defined by $\mathtt{Lab}(\delta) = 1$ and $\mathtt{Lab}(\delta') = \mathtt{Lab}(\delta'') = 0$. The labelled graph

$(G, \texttt{Lab})$ is depicted on the left of figure 1, it is right-resolving and its sofic shift is composed by the bi-infinite words such that the number of 0 between every two consecutive 1 is even.

Note that vertices without incoming edges or without outgoing edges cannot be visited by a bi-infinite words and can hence be deleted without loss of generality. The resulted graph is called pruned.

2.5 Comparison with finite state automata

A *non-deterministic finite state automaton* (NFA) $B = (Q, \Delta, \texttt{Lab}, I, F)$ is a labelled graph (with vertices called states) augmented with sets of initial and final states I and F. The set of words that label paths leading from initial states to final states is the language of B (such language is called *regular*). One can see that sets of allowed factors $\cup_{n\in\mathbb{N}}[A]_n$ of sofic shifts A correspond to regular languages L that are *factorial* (every factor of a word of L is a word of L), *left-* and *right-extensible* (if $w \in L$ then there exists $a, b \in \Sigma$ such that $aw \in L$ and $wb \in L$). Being factorial and extensible are suitable properties in the classical context of constrained-channel coding (for which symbolic dynamics offer a meaningful framework). We already motivated these properties for timed languages in [2] where we built the basis of a timed theory of channel coding.

There exists a variation of the theory of symbolic dynamics based on mono-infinite words rather than bi-infinite words, that is, indexed by $\mathbb{N}$ rather than $\mathbb{Z}$ (see §13.8 of [19]). The "one-sided" shift spaces of this theory are exactly the omega-regular languages recognised by Büchi automata with all states initial and final.

3 Factor based characterisations

In the previous section, we gave topological definitions of shift spaces, their entropies and morphisms. Simpler characterisations of these objects based on factors are available in symbolic dynamics (i.e. when the alphabet is finite). In this section we generalise these characterisations to general alphabet shift spaces. We carefully replace properties that implicitly use finite cardinality of sets in symbolic dynamics by similar properties involving compactness or finite measure of corresponding sets in our more general setting.

3.1 Factor based characterisation of general alphabet shift spaces

We recall that the alphabet $\mathcal{C}$ considered in the following is a compact metric space.

Definition 1. *Given a family $O = (O_n)_{n\in\mathbb{N}\setminus\{0\}}$ where all O_n are open sets of $\mathcal{C}^n$, we denote by $\mathcal{F}(O)$ the set of bi-infinite words not having factors in O :* $\mathcal{F}(O) = \{x \in \mathcal{C}^{\mathbb{Z}} \mid \forall i \in \mathbb{Z}, \forall n \in \mathbb{N} \setminus \{0\}, x_{[i..i+n-1]} \notin O_n\}$.

We have also the dual definition

Definition 2. *Given a family* $F = (F_n)_{n\in\mathbb{N}\setminus\{0\}}$ *where all* F_n *are closed sets of* $\mathcal{C}^n$*, we denote by* $\mathcal{B}(F)$ *the set of bi-infinite words whose allowed factors are those of* F : $\mathcal{B}(F) = \{x \in \mathcal{C}^{\mathbb{Z}} \mid \forall i \in \mathbb{Z}, \forall n \in \mathbb{N}\setminus\{0\}, x_{[i..i+n-1]} \in F_n\}$

Theorem 1. *A subset of* $\mathcal{C}^{\mathbb{Z}}$ *is a shift space iff it can be defined as a* $\mathcal{F}(O)$ *iff it can be defined as a* $\mathcal{B}(F)$*.*

Note that, in symbolic dynamics (for which $\mathcal{C}$ is finite), there is no need of specifying which sets are open or closed as all finite sets satisfy both properties.

Example 2. We introduce five running examples (indexed with roman number).

Let $\mathcal{C}^{\mathtt{I}} \stackrel{\text{def}}{=} [0,1] \times \{a\}$ and the set of forbidden factors be given by $O_2^{\mathtt{I}} \stackrel{\text{def}}{=} \{(t,a)(t',a) \mid t+t' > 1\}$. The shift space $X^{\mathtt{I}} \stackrel{\text{def}}{=} \mathcal{F}(O^{\mathtt{I}})$ is the set $\{(t_i,a)_{i\in\mathbb{Z}} \mid t_i + t_{i+1} \le 1\}$.

Let $\mathcal{C}^{\mathtt{II}} \stackrel{\text{def}}{=} [0,1] \times \{a\}$ and $O_1^{\mathtt{II}} \stackrel{\text{def}}{=} \{(t,a) \mid t < 1\}$. The only element of the shift space $X^{\mathtt{II}} \stackrel{\text{def}}{=} \mathcal{F}(O^{\mathtt{II}})$ is $(1,a)^{\mathbb{Z}}$.

Let $\mathcal{C}^{\mathtt{III}} \stackrel{\text{def}}{=} [0,1] \times \{a,b\}$ and[2] $O_2^{\mathtt{III}} \stackrel{\text{def}}{=} \{(t,a)(t',b) \mid t < 1, t' \in [0,1]\} \cup \{(t,l)(t',l) \mid l \in \{a,b\},\ t,t' \in [0,1]\}$. The shift space $X^{\mathtt{III}} \stackrel{\text{def}}{=} \mathcal{F}(O^{\mathtt{III}})$ is the set of bi-infinite words of the form $[(t_i,a_i)(t_{i+1},b_{i+1})]_{i\in 2\mathbb{Z}}$ or $[(t_i,a_i)(t_{i+1},b_{i+1})]_{i\in 2\mathbb{Z}+1}$ with $t_i = 1$ and $t_{i+1} \in [0,1]$.

Let $\mathcal{C}^{\mathtt{IV}} \stackrel{\text{def}}{=} [0,1] \times \{b\}$ and $O_n^{\mathtt{IV}} \stackrel{\text{def}}{=} \{(t_1,b)\cdots(t_n,b) \mid t_1 + \ldots + t_n > 1\}$. The shift space $X^{\mathtt{IV}} \stackrel{\text{def}}{=} \mathcal{F}(O^{\mathtt{IV}})$ is the set of bi-infinite words $(t_i,b)_{i\in\mathbb{Z}}$ satisfying the (bi-infinite) *Zeno condition* $\sum_{i\in\mathbb{Z}} t_i \le 1$.

Let $\mathcal{C}^{\mathtt{V}} \stackrel{\text{def}}{=} [0,1] \times \{a,b\}$ and $O_1^{\mathtt{V}} \stackrel{\text{def}}{=} \{(t,a) \mid t < 1\}$ and $O_n^{\mathtt{V}} \stackrel{\text{def}}{=} \{(t_1,b)\cdots(t_n,b) \mid t_1 + \ldots + t_n > 1\}$. Every bi-infinite words of $X^{\mathtt{V}} \stackrel{\text{def}}{=} \mathcal{F}(O^{\mathtt{V}})$ has its delays corresponding to events a equal to 1 (as for $X^{\mathtt{II}}$) and the sum of delays of blocks of consecutive b bounded by 1 (as for $X^{\mathtt{IV}}$).

3.2 Entropies for general alphabet shift spaces

Topological entropy is very useful to compare dynamical systems. Unfortunately it is infinite for shift spaces on infinite alphabet as remarked in [20].

Fact 1 *Let* $\mathcal{C}$ *be an infinite compact metric space then* $h_{\text{top}}(\mathcal{C}^{\mathbb{Z}}) = +\infty$*.*

A first approach to circumvent this issue is to generalise the following characterisation of the entropy that holds for finite alphabet shift space X (see [19]),

$$h_{\text{top}}(X) = \lim_{n\to+\infty} \frac{1}{n} \log_2 |X_n|. \tag{2}$$

Asarin and Degorre replaced cardinality measures by volume measures (explained below) to define an ad hoc notion of entropy for timed automata in

[2] We recall that a set A is open in $[0,1]$ if it is of the form $A = B \cap [0,1]$ with B an open set of $\mathbb{R}$ (i.e. a union of open intervals). In particular $[0,1]$ is open in $[0,1]$.

[5] called volumetric in later papers [2,3,4]. Here, we describe both entropies (classical and volumetric) in a unified and more general framework.

The compact metric spaces $\mathcal{C}$ considered in this paper are endowed with a "natural" measure μ and hence the set $\mathcal{C}^n$ has the product measure μ^n. For example the measure on Σ^n, for finite Σ, is the counting measure; the measure on $[0,M]^n$ is the n-dimensional volume (aka. Lebesgue measure); and the measure on $([0,M]\times\Sigma)^n \cong [0,M]^n \times \Sigma^n$ also called volume is the product of the two preceding measures. More precisely, a subset E of $([0,M]\times\Sigma)^n$ can be seen as a formal sum of subsets $E^{|w} \subseteq [0,M]^n$ associated with $w \in \Sigma^n$ as follows $E^{|w} = \{(t_1,\ldots,t_n) \mid (t_1,w_1)\ldots(t_n,w_n) \in E\}$. The volume of E is just the sum of the volumes of $E^{|w}$: $\mathtt{Vol}(E) = \sum_{w\in\Sigma^n} \mathtt{Vol}(E^{|w})$.

We now give our general definition of entropy for general alphabet shift spaces:

Definition 3. *Given a compact metric space $\mathcal{C}$ and a measure μ on it, the entropy of a subshift $X \subseteq \mathcal{C}^{\mathbb{Z}}$ is*

$$\mathcal{H}(X) = \lim_{n\to+\infty} \frac{1}{n} \log_2 \mu^n(X_n) \quad \textit{(with } X_n \stackrel{\text{def}}{=} \{x_{[i+1..i+n]} \mid x \in X,\ i \in \mathbb{Z}\}\textit{).} \quad (3)$$

Applying Fekete's lemma ([16]) on sub-additive sequence to $(\log_2 \mu^n(X_n))_{n\in\mathbb{N}}$ ensures that the limit exists in $\mathbb{R}\cup\{-\infty,+\infty\}$.

Another way of circumventing the problem of the infinite topological entropy is to consider an asymptotic expansion of the ε-entropy instead of its limit when ε tends to 0 in Equation (1). This has been done fruitfully for volumetric entropy of timed automata in [12] (recalled in Theorem 7 below).

3.3 Sliding block codes for general alphabet shift spaces

In this section $\mathcal{C}$ and $\mathcal{C}'$ denote two compact metric spaces, X and Y denote subshifts of $\mathcal{C}^{\mathbb{Z}}$ and $\mathcal{C}'^{\mathbb{Z}}$ respectively. Given a function ψ from X to $\mathcal{C}'$ we denote by $\psi^\infty : X \to \mathcal{C}'^{\mathbb{Z}}$ the function defined by $(\psi^\infty(x))_i = \psi(\sigma^i(x))$. Such functions are those that commute with the shifts (i.e. $\sigma\circ\phi = \phi\circ\sigma$) and are thus morphisms if and only if continuous.

We say that ψ is a $(2m+1)$-block function when for every x, $\psi(x)$ depends only on the $(2m+1)$-central factor $x_{[-m..m]}$, i.e. there exists a function $f : \mathcal{C}^{2m+1} \to \mathcal{C}'$ such that for every x, $\psi(x) = f(x_{[-m..m]})$. One can remark that ψ is continuous iff so is f. A function ϕ that is equal to some ψ^∞ with ψ a (continuous) block function is called a *(continuous) sliding block code*.

The following famous theorem gives a characterization of the morphisms of finite-alphabet shift spaces as sliding block codes.

Theorem 2 (Curtis-Hedlund-Lyndon). *Let X and Y be two finite alphabet shift spaces. A function $\varphi : X \to Y$ is a morphism if and only if it is a sliding block code.*

This Theorem cannot be extended to the case of general alphabets shift spaces as highlighted by the following fact:

Fact 2 *There are endomorphisms of* $[0,1]^{\mathbb{Z}}$ *which are not sliding block codes.*

Proof (Sketch). Let $\psi : [0,1]^{\mathbb{Z}} \to [0,1]$ be defined by $\psi(x) = \frac{1}{3}\sum_{i\in\mathbb{Z}} \frac{x_i}{2^{|i|}}$. One can show that ψ^{∞} is an endomorphism of $[0,1]^{\mathbb{Z}}$ which is not a sliding block code.

An adapted version of Theorem 2 can however be stated as follows.

Theorem 3. *Every morphism* ϕ *from a shift space* X *to a full shift* $\mathcal{C}^{\mathbb{Z}}$ *is the uniform limit of a sequence of continuous sliding block codes* $(\phi_m)_{m\in\mathbb{N}}$ *from* X *to* $\mathcal{C}^{\mathbb{Z}}$, *that is* $\sup_{x\in X} \overline{d}(\phi(x), \phi_m(x)) \to_{m\to+\infty} 0$ *where* $\overline{d}$ *is the metric on* $\mathcal{C}^{\mathbb{Z}}$.

Proof (Sketch). We first characterise every morphism as a function of the form ψ^{∞} with ψ a continuous function from X to $\mathcal{C}$. We then show that every continuous function from X to $\mathcal{C}$ is a uniform limit of continuous block functions from X to $\mathcal{C}$. Theorem 3 can then be proved by combining these two last results.

4 Timed shift spaces and their measures

In this section we define and study timed sofic shifts which are a way to see regular timed languages [1] as (general alphabet) shift spaces.

4.1 Timed shift spaces

Timed graphs. Informally, a timed graph is to a timed automaton what a graph is to an automaton: an automaton without initial, final states as well as labels on transitions. Formally, a *timed graph* (TG) is a tuple $\mathcal{G} = (C, Q, \Delta)$ such that

- C is a finite set of bounded *clocks* which are variables ranging over $[0, M]$ with $M \in \mathbb{N}$;
- Q is a finite set of *locations*;
- Δ is a finite set of *transitions*. Any transition $\delta \in \Delta$ has an *origin* $\delta^- \in Q$; a *destination* $\delta^+ \in Q$; a *closed guard* $\mathfrak{g}_\delta$, that is a conjunction of inequalities of the form $x \sim c$ or $x \sim y + c$, where x and y are clocks, $\sim \in \{\leq, =, \geq\}$ and $c \in \{0, \dots, M\}$; and a *reset* function $\mathfrak{r}_\delta$ determined by a subset of clocks $B \subseteq C$: it resets to 0 all the clocks in B and does not modify the value of the other clocks.

States, timed transitions and successor actions. We denote by $\mathbb{S}$ the set of *states* which are couples of a location and a clock vector $\mathbb{S} \stackrel{\text{def}}{=} Q \times [0, M]^C$. A *timed transition* is an element (t, δ) of $\mathbb{A} \stackrel{\text{def}}{=} [0, M] \times \Delta$. The *time delay* t represents the time before firing the transition δ.

Given a state $s = (q, \boldsymbol{x}) \in \mathbb{S}$ and a timed transition $\alpha = (t, \delta) \in \mathbb{A}$ the *successor* of s by α is denoted by $s \triangleright \alpha$ and defined as follows. If $\delta^- = q$ and $\boldsymbol{x} + t$ satisfies the guard $\mathfrak{g}_\delta$ then $s \triangleright \alpha = (\delta^+, \mathfrak{r}_\delta(\boldsymbol{x} + t))$ else $s \triangleright \alpha = \perp$. Here and in the rest of the paper $\perp$ represents undefined states.

Runs and their shifts. A *bi-infinite run* of a timed graph $\mathcal{G}$ is a bi-infinite word $(s_i, \alpha_i)_{i\in\mathbb{Z}} \in (\mathbb{S} \times \mathbb{A})^{\mathbb{Z}}$ such that $s_{i+1} = s_i \triangleright \alpha_i \neq \bot$ for all $i \in \mathbb{Z}$. Consider the timed graph $\mathcal{G}^{\mathtt{I}}$ depicted on Figure 2 (left) and a bi-infinte words whose ith letter is $\alpha_i = (t_i, \delta_i)$ with $\delta_i = \delta$ if i is even and $\delta_i = \delta'$ otherwise and such that $t_i + t_{i+1} \in [0, 1]$ for every $i \in \mathbb{Z}$; define for even i, $s_i = (p, (t_{i-1}, 0))$ and $s_{i+1} = (q, (0, t_i))$ then $(s_i, \alpha_i)_{i\in\mathbb{Z}}$ is a run of $\mathcal{G}^{\mathtt{I}}$.

Proposition 1. *The set of bi-infinite runs of $\mathcal{G}$ is a sub-shift of $(\mathbb{S} \times \mathbb{A})^{\mathbb{Z}}$.*

Timed edge shift and timed sofic shift. We are now ready to define the timed generalisation of edge shift and sofic shift.

Proposition-definition 1 (Timed edge shift) *The following set is a sub-shift of $\mathbb{A}^{\mathbb{Z}}$ called the* timed edge shift *of $\mathcal{G}$ and denoted by $[\mathcal{G}]$:*

$$[\mathcal{G}] = \{(\alpha_i)_{i\in\mathbb{Z}} \mid \exists (s_i)_{i\in\mathbb{Z}} \in \mathbb{S}^{\mathbb{Z}},\ \forall i \in \mathbb{Z}, s_{i+1} = s_i \triangleright \alpha_i\}$$

When adding to a TG $\mathcal{G}$ a labelling function $\mathtt{Lab} : \Delta \to \Sigma$ from the set of transition Δ to a finite alphabet of event Σ we obtain a *labelled timed graph* (LTG) $\mathcal{A} = (\mathcal{G}, \mathtt{Lab})$. Abusing the notation we extend the labelling function to timed transitions and runs as follows: $\mathtt{Lab}(\alpha) = (t, \mathtt{Lab}(\delta))$ when $\alpha = (t, \delta)$ and $\mathtt{Lab}\left((s_i, \alpha_i)_{i\in\mathbb{Z}}\right) = (s_i, \mathtt{Lab}(\alpha_i))_{i\in\mathbb{Z}}$. Thus we use two kinds of *timed alphabet*: alphabet of timed transitions $\mathbb{A} = [0, M] \times \Delta$ and alphabets of *timed letters* $\mathtt{Lab}(\mathbb{A}) = [0, M] \times \mathtt{Lab}(\Delta)$.

Proposition-definition 2 (Timed sofic shift) *Let $\mathcal{A} = (\mathcal{G}, \mathtt{Lab})$ be a labelled closed timed graph then the set $[\mathcal{A}] = \{\mathtt{Lab}\left((\alpha_i)_{i\in\mathbb{Z}}\right) \mid (\alpha_i)_{i\in\mathbb{Z}} \in [\mathcal{G}]\}$ is a sub-shift of $(\mathtt{Lab}(\mathbb{A}))^{\mathbb{Z}}$ called* the timed sofic shift *of $\mathcal{A}$.*

An LTG is called *right resolving* if every two different transitions labelled by the same letter and starting from the same location have pairwise incompatible guards. As for classical symbolic dynamics, being right-resolving is the same thing as being deterministic less the property of having a unique initial state (see [1] for the usual definition of determinism in timed automata context). The LTGs of Figure 2 are right-resolving, they recognise the shift spaces of Example 2.

4.2 Discretisation of shift spaces and their entropy

Several definitions of ε-entropy for compact metric alphabet shift spaces were recalled in Section 2.2. Here, we give a simpler definition of ε-entropy for timed shift spaces which is asymptotically linked to the other ε-entropies in Proposition 3. This new definition of ε-entropy is based on discretisation of the timed shift space we explore now.

We call ε-*discrete* the different objects involving delays and clocks multiple of ε (i.e vector of delays of $\mathbb{R}^n$, timed words, bi-infinite timed words, runs, etc.). The

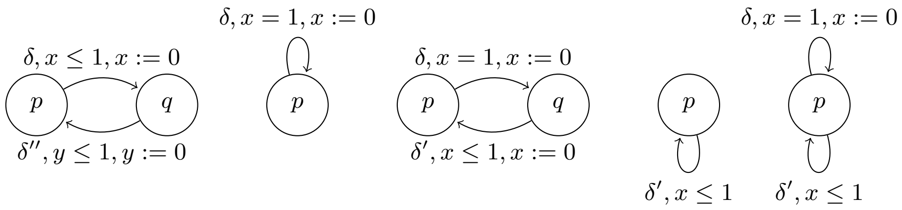

Fig. 2. From left to right LTGs $\mathcal{A}^i = (\mathcal{G}^i, \texttt{Lab})$ for $i = \texttt{I..V}$ with $\texttt{Lab}(\delta) = \texttt{Lab}(\delta'') = a$ and $\texttt{Lab}(\delta') = b$. They recognise timed sofic shift of Example 2.

ε-*discretisation* of a set B denoted by B_ε is the set of its ε-discrete elements. For instance, for $\mathbb{A} = [0, M] \times \Delta$, $\mathbb{A}_\varepsilon = \{0, \varepsilon, \dots, M\} \times \Delta$; for $X \subseteq \mathbb{A}^\mathbb{Z}$, $X_\varepsilon = X \cap \mathbb{A}_\varepsilon^\mathbb{Z}$.

The following proposition states a discretisation of timed shift spaces, the resulting shift space being a finite alphabet shift space.

Proposition 2. *If X is a sub-shift of $\mathbb{B}^\mathbb{Z}$ where $\mathbb{B}$ is a timed alphabet, then X_ε is a sub-shift of $\mathbb{B}_\varepsilon^\mathbb{Z}$.*

We define the ε-entropy of a shift $X \subseteq \mathbb{B}^\mathbb{Z}$ as the entropy of the shift $X_\varepsilon \subseteq \mathbb{B}_\varepsilon^\mathbb{Z}$:

$$h_\varepsilon(X) \stackrel{\text{def}}{=} h_{\text{top}}(X_\varepsilon) = \lim_{n \to \infty} \frac{1}{n} \log_2 |X_{\varepsilon,n}|.$$

Proposition 3. *For every two discretisation steps $\varepsilon' \geq \varepsilon$, for every timed sofic shift X, it holds that $h_{2\varepsilon'}(X) \leq h_{2\varepsilon}^S(X) \leq h_\varepsilon^N(X) \leq h_\varepsilon(X)$.*

Let $\mathcal{A} = (\mathcal{G}, \texttt{Lab})$ be a right-resolving LTG. The discretisation of the timed sofic shift $[\mathcal{A}]$ is the sofic shift of a right-resolving finite labelled graph $\mathcal{A}_\varepsilon$ obtained from $\mathcal{A}$ by a discretisation of its timed transitions and states as follows: $\mathcal{A}_\varepsilon = ((Q_\varepsilon, \Delta'), \texttt{Lab}')$ with $Q_\varepsilon = \mathbb{S} \cap (Q \times \{0, \varepsilon, \dots, M\}^d)$, $\texttt{Lab}' : \Delta' \to \texttt{Lab}(\mathbb{A}_\varepsilon)$ and there is a transition $\delta_{s,\alpha,s'} \in \Delta'$ going from ${\delta_{s,\alpha,s'}}^- \stackrel{\text{def}}{=} s$ to ${\delta_{s,\alpha,s'}}^+ \stackrel{\text{def}}{=} s'$ and labelled by $\texttt{Lab}'(\delta_{s,\alpha,s'}) \stackrel{\text{def}}{=} \texttt{Lab}(\alpha)$ iff $s \triangleright \alpha = s'$.

Proposition 4. *Let $\mathcal{A}$ be a right-resolving LTG, then $[\mathcal{A}_\varepsilon] = [\mathcal{A}]_\varepsilon$.*

As a corollary the computation of the ε-entropy of a timed sofic shift reduces to the computation of the entropy of a (finite alphabet) sofic shift.

Corollary 1. *Let $\mathcal{A} = (\mathcal{G}, \texttt{Lab})$ be a right-resolving LTG, its ε-entropy is the topological entropy of the sofic shift of $\mathcal{A}_\varepsilon$: $h_\varepsilon([\mathcal{A}]) = h_{\text{top}}([\mathcal{A}_\varepsilon])$. In particular, $h_\varepsilon([\mathcal{A}])$ can be computed as the logarithm of the spectral radius of the adjacency matrix of $\mathcal{A}_\varepsilon$ (This matrix has order $O(|Q|/\varepsilon^{|C|})$ where C is the set of clocks).*

In [6], a similar approach was used to over- and under-approximate the quantity $\mathcal{H}(\mathcal{A}) + \log_2(1/\varepsilon)$ for a timed automaton $\mathcal{A}$ without guarantee of convergence. The asymptotic equality of this quantity with the ε-entropy was later proved in [12] (Theorem 7 below).

The following theorem justifies that one can focus on TGs rather than right-resolving LTGs without loss of generality for the entropies.

Theorem 4. *Let $\mathcal{A} = (\mathcal{G}, \mathtt{Lab})$ be a right-resolving LTG, then $\mathcal{H}([\mathcal{A}]) = \mathcal{H}([\mathcal{G}])$ and $h_\varepsilon([\mathcal{A}]) = h_\varepsilon([\mathcal{G}])$.*

Proof (Sketch). We use the following chain of equalities $h_\varepsilon([\mathcal{A}]) = h_{\text{top}}([\mathcal{A}_\varepsilon]) = h_{\text{top}}([\mathcal{G}_\varepsilon]) = h_\varepsilon([\mathcal{G}])$ where the first and third equalities are given by Corollary 1 and the second equality follows from a classical correspondence of entropy between finite alphabet sofic shift and their underlying edge shift (see [19]).

To prove that $\mathcal{H}([\mathcal{A}]) \leq \mathcal{H}([\mathcal{G}])$, we use the fact that for every word $w \in \Sigma^*$, $[\mathcal{A}]_n^{|w} = \cup_{\pi \in \mathtt{Lab}^{-1}(w)} [\mathcal{G}]_n^{|\pi}$, pass to volumes: $\mathtt{Vol}([\mathcal{A}]_n^{|w}) \leq \sum_{\pi \in \mathtt{Lab}^{-1}(w)} \mathtt{Vol}([\mathcal{G}]_n^{|\pi})$ and apply operator $\lim_{n\to\infty} \frac{1}{n} \log_2 \sum_{w \in \Sigma^n} (\ )$. The converse inequality $\mathcal{H}([\mathcal{A}]) \geq \mathcal{H}([\mathcal{G}])$ is more involved since the sets $[\mathcal{G}]_n^{|\pi}$ for $\pi \in \mathtt{Lab}^{-1}(w)$ can overlap. However, using the fact that $\mathcal{A} = (\mathcal{G}, \mathtt{Lab})$ is right-resolving, one can show that the volume of the overlap does not contribute to the entropy as it decreases too fast when $n \to \infty$ and hence $\mathcal{H}([\mathcal{A}]) \geq \mathcal{H}([\mathcal{G}])$ also holds. □

4.3 Metric mean dimension of timed sofic shifts

Given a timed graph $\mathcal{G}$, a path π and $n = |\pi|$, it is well known that the set of delay vectors $[\mathcal{G}]_n^{|\pi} \stackrel{\text{def}}{=} \{(t_1, \ldots, t_n) \mid (t_1, \pi_1) \ldots (t_n, \pi_n) \in [\mathcal{G}]_n\}$ is a polytope. One can define the dimension $\dim(\pi)$ of a path, as the affine dimension of its polytope, that is, the maximal number of affinely independent points in the polytope minus 1 (see also [8]). A TG $\mathcal{G}$ is *fleshy* whenever all its path are full dimensional, that is $\dim(\pi) = |\pi|$ for every paths π of $\mathcal{G}$. It can happen that when the length of paths considered tends to infinity the delays are more and more constrained resulting in a null average choice. This kind of phenomena is measured by metric mean dimension defined and illustrated on several examples below.

The *metric mean dimension* [20] of a dynamical system $((X, d), f)$ is:

$$\mathbf{mdim}(X) = \liminf_{\varepsilon\to 0} \frac{\log_2 h_\varepsilon^S(X)}{\log_2(1/\varepsilon)} = \liminf_{\varepsilon\to 0} \frac{\log_2 h_\varepsilon^N(X)}{\log_2(1/\varepsilon)}$$

The second equality is due to Lemma 1. As a corollary of Proposition 3, we can characterise the metric mean dimension of timed sofic shift explicitly in terms of their ε-entropy as follows:

Corollary 2. *The metric mean dimension of a timed sofic shift X is:*

$$\mathbf{mdim}(X) = \liminf_{\varepsilon\to 0} \frac{\log_2 h_\varepsilon(X)}{\log_2(1/\varepsilon)} \tag{4}$$

Note that if $X \subseteq Y$ then $\mathbf{mdim}(X) \leq \mathbf{mdim}(Y)$ and that $\mathbf{mdim}(\mathbb{B}^{\mathbb{Z}}) = 1$ for every timed alphabet $\mathbb{B}$. Thus $\mathbf{mdim}(X) \leq 1$ for every sub-shift X of $\mathbb{B}^{\mathbb{Z}}$.

Example 3 (Example 2 and Figure 2 continued). The shift space $X^{\mathtt{I}}$ has metric mean dimension 1 since all the delays can be chosen independently in the interval $[0, 1/2]$. The shift space $X^{\mathtt{II}}$ contains only one element and has thus metric mean dimension 0. The metric mean dimension of $X^{\mathtt{III}}$ is $1/2$. This corresponds to the intuition that a full choice can be made half of the time, since delays before edges δ are always in the 0-dimensional singleton $\{1\}$ while delays before edges δ' are to be chosen in the 1-dimensional interval $[0, 1]$. The number of ε-discrete points in the polytope $[\mathcal{G}^{\mathtt{IV}}]_n^{|\pi}$ for the only path π of length n is $\binom{n+1/\varepsilon}{n}$ (Lemma 2 below) and thus the metric mean dimension of $X^{\mathtt{IV}}$ is null. Intuitively, there are less and less choices as n increases. Every path of $\mathcal{G}^{\mathtt{V}}$ containing a δ yields a volume 0, the only path of length n that yields a non-null volume is δ'^n. This volume is $\mathtt{Vol}(X_n^{\mathtt{V}}) = 1/n!$ and hence the entropy is $\mathcal{H}(X^{\mathtt{V}}) = -\infty$. The metric mean dimension of $X^{\mathtt{V}}$ is 1. Indeed, for every positive integer m, the paths in $(\delta^{m-1}\delta')^*$ yield a metric mean dimension equal to $(m-1)/m$ and thus $\mathbf{mdim} \geq 1 - 1/m$ for every $m > 0$.

Lemma 2 (Few points in a simplex). *The number of ε-discrete points in a simplex described by inequalities $0 \leq u_1 \leq \cdots \leq u_n \leq M$ (resp. by inequalities $\sum_{i=1}^n u_i \leq M$ and $u_i \geq 0$) is $\binom{n+M/\varepsilon}{n}$ and $(1/n)\log_2\left[\binom{n+M/\varepsilon}{n}\right] \to_{n\to+\infty} 0$.*

One can generalise $\mathcal{G}^{\mathtt{III}}$ by defining a cycle with k transition b and $l-k$ transition a for every naturals $1 \leq k \leq l$. The resulting timed sofic shift has metric mean dimension k/l. More surprisingly arbitrary rational metric mean dimensions lower than 1 can be obtained from timed graph that have full dimensional sets of factors X_n for every length n.

Theorem 5. *For every rational $r \in \mathbb{Q} \cap [0, 1]$, there exists a timed sofic shift X recognised by a right-resolving fleshy LTG such that $\mathbf{mdim}(X) = r$.*

Proof (sketch). Examples of fleshy timed graph with metric mean dimension 0 or 1 have already been treated ($\mathcal{G}^{\mathtt{I}}$ and $\mathcal{G}^{\mathtt{II}}$). For every $a, b \in \mathbb{N}$ such that $0 < a < b$, we describe a cyclic timed graph with $2b$ edges and metric mean dimension $a/b \in \mathbb{Q} \cap (0, 1)$. The edges are $q_i \xrightarrow{x\in[i,i+1]} q_{i+1}$ for $i = 0..2a-1$; $q_i \xrightarrow{y\geq 2a+1, x\in[2a,2a+1]} q_{i+1}$ for $i = 2a..2b-3$; $q_{2b-2} \xrightarrow{y\geq 2a+1, x\in[2a,2a+1], y:=0} q_{2b-1}$; $q_{2b-1} \xrightarrow{x\in[2a,2a+1], x:=0} q_0$. One can see that each edge of the first form yields a full dimension ($i = 0..2a-1$). The other edges impose stringent constraints on clocks and delays like in a simplex. This yields a null mean dimension for these edges (Lemma 2). At the end for each cycle of length $2b$ there are $2a$ full dimensional edges and thus $\mathbf{mdim} = 2a/2b = a/b$.

The thick timed sofic shifts. In [12], we characterised precisely a dichotomy between *thin* and *thick* timed automata based on entropy, the former having infinite entropy ($\mathcal{H} = -\infty$) while the latter having a finite one ($\mathcal{H} > -\infty$). Beyond its entropy based definition we argued that this dichotomy is between bad behaving and well behaving TA. The former are in some weak sense Zeno, are non robust against clock perturbations, cannot be discretised well, etc. while the latter enjoy better properties such as a good discretisation, a quantitative pumping lemma and the existence of so-called forgetful cycles.

The metric mean dimension measurement gives a novel characterisation of thickness in terms of maximal metric mean dimension:

Theorem 6. *For timed sofic shifts X recognised by fleshy LTGs, thickness is equivalent to maximal metric mean dimension: $\mathcal{H}(X) > -\infty$ iff $\mathbf{mdim}(X) = 1$.*

Note that X^{V} satisfies both $\mathcal{H}(X^{\mathsf{V}}) = -\infty$ and $\mathbf{mdim}(X^{\mathsf{V}}) = 1$. This means that fleshiness is necessary in Theorem 6. Remark that regarding thickness, fleshiness is assumed wlog. since pruning the transitions involving punctual guards (e.g. $x = 1$) does not change the volume nor the entropy.

Beyond the pure dichotomy between thin and thick timed languages [12], Theorem 5 and 6 provide a deeper insight of convergence phenomena among thin timed languages. There is a whole continuum of thin timed languages between the extremely narrow ones of metric mean dimension 0 where all delays of timed words are constrained in a very stringent way, and the ones of dimension almost 1 for which full freedom in the delay is available at almost each transition.

For the sake of completness we recall one of the main theorems of our previous work [12] in terms of timed symbolic dynamics. This theorem ensures that the approximation of the entropy by discretisation initiated in [6] converges.

Theorem 7 (A symbolic dynamics version of Theorem 4 of [12]). *Let $\mathcal{A}$ be a right-resolving fleshy thick LTG then its volumetric entropy can be approximated by its ε-entropy as follows: $h_\varepsilon([\mathcal{A}]) = \log_2(1/\varepsilon) + \mathcal{H}([\mathcal{A}]) + o(1)$.*

One can interpret as in [6,2] $\mathcal{H}([\mathcal{A}])$ as the average information per event and $\log_2(1/\varepsilon)$ as the information necessary to represents with precision ε the time between two events.

5 Conclusion and perspectives

In this paper, we introduced a theory of timed symbolic dynamics. We revisited previous works on volumetry of timed languages [5,6,12] within this new theory. We adapted to timed sofic shifts the metric mean dimension of Lindenstrauss, Weiss and Gromov [20]. We also stated a generalisation of the Curtis-Hedlund-Lyndon theorem for shift spaces on alphabets that are compact metric spaces.

Fundamental objects of classical symbolic dynamics are so-called shifts of finite types (SFT): the shift spaces that can be defined with a finite set of forbidden factors. In fact, such shifts are conjugated to edge shifts. That is why we are able to lift results from classical symbolic dynamics to the timed case without referring to SFTs (but referring to graphs and edge shifts). The entropy of probability measures on shift spaces is well studied in symbolic dynamics.

The entropy of a probability measure on an edge shift (or equivalently SFT) is always bounded from above by the entropy of its underlying graph. An important result obtained by Parry [23] is that for every edge shift associated to a strongly connected graph, there is a unique probability measure whose entropy is equal to that of the edge shift. This probability measure is given by a Markov chain on the graph of the edge shift (originally described by Shannon [25]). We already generalised such a Shannon-Parry Markov chain to the timed settings motivated by verification purposes [9,11]. However, we left open the question of uniqueness. Symbolic dynamics techniques (as those of Parry) could hence be useful to address this problem.

In [7], Asarin and Degorre introduced a mean dimension (that we call *syntactic*) for timed automata and proposed an algebraic characterisation of it. However this dimension only measures the proportion of non-punctual transitions along runs but not the Zeno behaviours. For instance, every fleshy timed graph has syntactic mean dimension 1 including $\mathcal{G}^{\mathtt{IV}}$. It seems easy to show that the syntactic mean dimension is upper bounded by the metric mean dimension. The case of equality is more involved and still needs to be investigated.

Metric mean dimension and controllability. In robust control [24,22], the goal is to design a controller that chooses step by step an infinite timed word satisfying a Büchi condition even if every delay is slightly perturbed. As every transition can be perturbed, the part that is robustly controllable does not contain punctual guards (fleshiness). Robust controllability is equivalent to the reachability (through fleshy transitions) of a forgetful cycle satisfying the Büchi condition [24,22]. It would be interesting to relax the condition that every transition must be robustly controllable and consider a framework where in some steps delays with arbitrary precision are chosen. In such a framework we would like to prove that timed automata with metric mean dimension α are the timed automata that can be robustly controlled with a frequency of α and that require arbitrary precision with frequency $1-\alpha$.

Acknowledgment. We gratefully acknowledge Eugène Asarin, Aldric Degorre and Dominique Perrin for motivating discussions.

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A Appendix

A.1 Proof of Theorem 1

Poof that X is a shift space $\Leftrightarrow$ it can be defined as a $\mathcal{B}(F)$

$\Rightarrow$) Let X be a shift space. Let $F = (X_n)_{n\in\mathbb{N}^*}$. By definition $X \subseteq \mathcal{B}(F)$. To show the converse inclusion, we take an $x \in \mathcal{B}(F)$ and prove that it belongs to X. As $x_{[-n..n]} \in X_{2n+1}$, there exists a bi-infinite word $x^n \in X$ such that $x^n_{[-n..n]} = x_{[-n..n]}$. The sequence $(x^n)_{n\in\mathbb{N}}$ converges to x. As this sequence takes its values in the closed set X, its limit x is also in X. It remains to prove that for all $n \in \mathbb{N}^*$, X_n is closed. It suffices to show that for every convergent sequence $(w^m)_{m\in\mathbb{N}}$ of X_n, its limit w belongs to X_n. For each n, there exists an $x^m \in X$ such that $x^m_{[0..n-1]} = w^m$. The sequence x^m of the compact $C^{\mathbb{Z}}$ admits a subsequence which converges toward an $x \in X$. We have $x_{[0..n-1]} = \lim_n x^m_{[0..n-1]} = \lim_n w^m = w$ and thus w belongs to X_n as a factor of $x \in X$. $\square$

$\Leftarrow$) Let $X = \mathcal{B}(F)$. X is shift invariant. It remains to show that X is closed. It suffices to show that for every convergent sequence $(x^m)_{m\in\mathbb{N}}$ of X, its limit x belongs to X. For all $n \in \mathbb{N}^*$, $m \in \mathbb{N}$ and $i \in \mathbb{Z}$, $x^m_{[i..i+n-1]} \in F_n$. As F_n is closed, $x_{[i..i+n-1]} = \lim x^m_{[i..i+n-1]}$ belongs to F_n. Every factor of x belongs to F thus $x \in \mathcal{B}(F) = X$. Hence X is closed. $\square$

Poof that X is a shift space $\Leftrightarrow$ it can be defined as a $\mathcal{F}(O)$

$\Rightarrow$) We take $F = (X_n)_{n\in\mathbb{N}^*}$ as above. We define $O = (\mathcal{C}^n \setminus X_n)_{n\in\mathbb{N}^*}$. For each n, O_n is relatively open in $\mathcal{C}^n$ since X_n is closed. By definition of O, $\mathcal{B}(F) = \mathcal{F}(0)$ which is equal to X. $\square$

$\Leftarrow$) Let $X = \mathcal{F}(O)$. X is shift invariant. Let us show that X is closed. It suffices to show that for every convergent sequence $(x^m)_{m\in\mathbb{N}}$ of X, its limit x belongs to X. For all $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, we have $x_{[i..i+n-1]} = \lim x^m_{[i..i+n-1]} \notin O^n$ since O^n is open. Thus $x \in \mathcal{F}(O)$. $\square$

A.2 Characterising N-ε-entropy and S-ε-entropy using sets of factors of a shift spaces.

The following proposition is needed in the proof of Proposition 3 but can be interesting by itself.

The distance $\overline{d}_n$ used in the definition of N-ε-entropy and S-ε-entropy are quite uneasy to deal with. We give here definitions of ε-entropies based on the product distances d^n defined on n-length words by $d^n(x_1 \cdots x_n, y_1 \cdots y_n) = \sup_{i=1..n} d(x_i, y_i)$. For this distance ε-balls are just hypercubes of side ε.

Proposition 5. *Let X be a compact metric alphabet shift space. The N-ε-entropy (resp S-ε-entropy) defined on bi-infinite words is equal to the following ε-entropy of X defined using finite factors:*

$$h^N_\varepsilon(X) = \limsup_{n\to\infty} \frac{1}{n} \log_2(\mathcal{N}_\varepsilon(X_n, d^n));$$

$$h^S_\varepsilon(X) = \limsup_{n\to\infty} \frac{1}{n} \log_2(\mathcal{S}_\varepsilon(X_n, d^n)).$$

This proposition is a consequence of the four following lemmas. In these Lemma we denote by $\mathtt{diam}(\mathcal{C})$ the diameter of $\mathcal{C}$, that is, $\mathtt{diam}(\mathcal{C}) \stackrel{\text{def}}{=} \max_{x,x' \in \mathcal{C}} d(x,x')$. This diameter exists and is finite due to the compactness of $\mathcal{C}$.

Lemma 3. *For all* $\varepsilon > 0 : \mathcal{N}_\varepsilon(X_n, d^n) \leq \mathcal{N}_\varepsilon(X, \overline{d}_n)$.

Proof. Let R be an ε-net of X for $\overline{d}_n$. We define $R' = \{x_{[0..n-1]} | \ x \in R\}$. We will show that R' is an ε-net of X_n. For all $y \in X_n$, there exists $y' \in X$ such that $y'_{[0..n-1]} = y$. There exists $x \in R$ such that $\overline{d}_n(x,y') \leq \varepsilon$. In particular, for all $i \in \{0,..,n-1\}, \overline{d}(\sigma^i(x), \sigma^i(y)) \leq \varepsilon$ thus $\max_{i\in\{0,..,n-1\}} d(x_i,y_i) < \varepsilon$ i.e $d^n(x,y) \leq \varepsilon$. □

Lemma 4. *For all* $\varepsilon > 0$*, there exists* l *such that* $\mathcal{N}_\varepsilon(X_{n+2l}, d^{n+2l}) \geq \mathcal{N}_\varepsilon(X, \overline{d}_n)$.

Proof. Let $l \in \mathbb{N}$ such that $\max_{|i|>l} \frac{\mathtt{diam}(\mathcal{C})}{2^{|i|}} \leq \varepsilon$. Let R_{n+2l} be an ε-net of X_{n+2l}. For all $x \in R_{n+2l}$, we choose $\hat{x} \in X$ such that $\hat{x}_{[-l..n+l-1]} = x$ and we define $\hat{R} = \{\hat{x} \mid x \in R\}$. We will show that $\hat{R}$ is an ε-net of X for $\overline{d}_n$. Let $y \in X$, there exists $x \in R_{n+2l}$ such that $d^{n+2l}(y_{[-l..n+l-1]}, x) \leq \varepsilon$. Therefore, for all $k \in \{0..n-1\} : \max_{i\in\{-l...l\}} \frac{d(x_{i+k},y_{i+k})}{2^{|i|}} \leq \varepsilon$ and thus $\sup_{i\in\mathbb{Z}} \frac{d(x_{i+k},y_{i+k})}{2^{|i|}} \leq \varepsilon$ i.e $\overline{d}_n(x,y) \leq \varepsilon$ □

Lemma 5. *For all* $\varepsilon > 0 : \mathcal{S}_\varepsilon(X_n, d^n) \leq \mathcal{S}_\varepsilon(X, \overline{d}_n)$.

Proof. Let S be an ε-net of X_n for d^n. For all $x \in S$, we choose $\hat{x} \in X$ such that $\hat{x}_{[0..n-1]} = x$ and we define $\hat{S} = \{\hat{x} \mid x \in S\}$. We have, for all $\hat{x}, \hat{y} \in \hat{S}$ $\overline{d}_n(\hat{x},\hat{y}) = \max_{k\in\{0...n-1\}} \sup_{i\in\mathbb{Z}} \frac{d(\hat{x}_{i+k},\hat{y}_{i+k})}{2^{|i|}} \geq \max_{k\in\{0..n-1\}} d(x_k,y_k) = d^n(x,y) > \varepsilon$. Thus $\hat{S}$ is ε-separated. □

Lemma 6. *For all* $\varepsilon > 0$*, there exists* l *such that* $\mathcal{S}_\varepsilon(X_{n+2l}, d^{n+2l}) \geq \mathcal{S}_\varepsilon(X, \overline{d}_n)$.

Proof. Let $l \in \mathbb{N}$ such that $\max_{|i|>l} \frac{\mathtt{diam}(\mathcal{C})}{2^{|i|}} \leq \varepsilon$. Let S a ε-net of X for $\overline{d}_n$. We define $S' = \{x_{[-l..l+n-1]} \mid x \in S\}$. We have, for all $x,y \in S$, $\overline{d}_n(x,y) = \max_{k\in\{0...n-1\}} \sup_{i\in\mathbb{Z}} \frac{d(x_{i+k},y_{i+k})}{2^{|i|}} > \varepsilon$. The terms of indices less than $-l$ and greater than $n+l-1$ are not taken into account as by definition of l they cannot be greater than ε. Therefore $d^{n+2l}(x_{[-l..l+n-1]}, y_{[-l..l+n-1]}) \geq \overline{d}_n(x,y) > \varepsilon$ and S' is ε-separated. □

A.3 Proof of Fact 2.

Before proving Fact 2 we characterise morphisms from X to $\mathcal{C}'^{\mathbb{Z}}$ where X and $\mathcal{C}'$ are defined as in Section 3.3.

We denote by $\mathcal{F}(X,\mathcal{C}')$ the function from X to $\mathcal{C}'$ and by $\mathcal{SC}(X,\mathcal{C}'^{\mathbb{Z}})$ the function from X to $\mathcal{C}'^{\mathbb{Z}}$ that commutes with the shift.

Lemma 7. *The mapping* $\psi \mapsto \psi^\infty$ *is a bijection from* $\mathcal{F}(X,\mathcal{C}')$ *to* $\mathcal{SC}(X,\mathcal{C}'^{\mathbb{Z}})$ *whose inverse is* $\phi \mapsto (x \mapsto \phi(x)_0)$*. Moreover* ψ *is continuous iff so is* ψ^∞*.*

Proof. By definition of ψ^∞ it holds that $\psi^\infty(x)_0 = \psi(x)$ and thus the two mappings defined above are mutually inverse. If ψ is continuous then for every sequence $(x^n)_{n\in\mathbb{N}}$ of elements of X, the convergence $x^n \mapsto_{n\to+\infty} x \in X$ implies that for every $i \in \mathbb{Z}$, $\psi^\infty(x^n)_i = \psi^\infty(\sigma^i(x^n)) \mapsto_{n\to+\infty} \psi^\infty(\sigma^i(x)) = \psi^\infty(x)_i$. This means that ψ^∞ is continuous when so is ψ. The converse is straightforward. □

Corollary 3. *Every morphism from X to $\mathcal{C}'^{\mathbb{Z}}$ is of the form ψ^∞ with ψ a continuous function from X to $\mathcal{C}'$.*

End of proof of Fact 2. Let $\psi : [0,1]^{\mathbb{Z}} \to [0,1]$ be defined by $\psi(x) = \frac{1}{3}\sum_{i\in\mathbb{Z}} \frac{x_i}{2^{|i|}}$. We show that ψ^∞ is an endomorphism of $[0,1]^{\mathbb{Z}}$ which is not a sliding block code. We show that ψ maps converging sequences to converging sequences and is thus continuous. Let $(x^k)_{k\in\mathbb{N}}$ be a sequence of bi-infinite words of $[0,1]^{\mathbb{Z}}$ that converges toward a bi-infinite word $x \in [0,1]^{\mathbb{Z}}$, that is $\sum_{i\in\mathbb{Z}} \frac{x_i^k}{2^{|i|}} \to_{k\to+\infty} \sum_{i\in\mathbb{Z}} \frac{x_i}{2^{|i|}}$. To this purpose we apply the dominate convergence theorem since $\frac{x_i^k}{2^{|i|}} \leq \frac{1}{2^{|i|}}$ for every $i \in \mathbb{N}$, $k \in \mathbb{N}$. Thus ψ is continuous and by virtue of Corollary 3 ψ^∞ is an endomorphism of $[0,1]^{\mathbb{Z}}$. □

A.4 proof of Theorem 3

We first state a similar result for functions from $\mathcal{C}^{\mathbb{Z}}$ to $\mathcal{C}'$.

Lemma 8. *Every continuous function from X to $\mathcal{C}'$ is a uniform limit of continuous block functions from X to $\mathcal{C}'$.*

Proof. Let $\psi \in \mathcal{F}(X, \mathcal{C}')$ be a continuous function. The mth *truncation* $f_m : x \mapsto x_{[-m..m]}$ is a continuous function from X to $\mathcal{C}^{2m+1}$. For $m \in \mathbb{N}$, let $g_m : X_{2m+1} \to X$ be a function such that for each w, $g_m(w)$ is an element of X with central factor w i.e. $g_m(w)_{[-m..m]} = w$. The function ψ_m is a $(2m+1)$-block continuous function. It remains to prove that $(\psi_m)_{m\in\mathbb{N}}$ converges toward ψ. As ψ is continuous between two compacts, it is also uniformly continuous, that is, for every arbitrary ε, there exists δ such that $\overline{d}(x,x') \leq \delta$ implies $d'(\psi(x),\psi(x')) \leq \varepsilon$ where d and d' denote the metric on $\mathcal{C}$ and $\mathcal{C}'$ respectively. We take $m \geq \log_2(\mathtt{diam}(\mathcal{C})/\delta) - 1$ so that for every $x \in \mathcal{C}^{\mathbb{Z}}$, $\overline{d}(g_m \circ f_m(x), x) \leq \mathtt{diam}(\mathcal{C})2^{-(m+1)} \leq \delta$ and thus $d_{\mathcal{C}'}(\psi_m(x),\psi(x)) \leq \varepsilon$. We are done the sequence $(\psi_m)_{m\in\mathbb{N}}$ of continuous block functions uniformly converges toward ψ. □

Now we can prove Theorem 3.

Proof. By Corollary 3, every morphism is of the form ψ^∞ with ψ a continuous function from X to $\mathcal{C}'$. By Lemma 8 just above there exists a sequence $(\psi_m)_{m\in\mathbb{N}}$ of continuous block functions that uniformly converges toward ψ. For every x

and i, the i^{th} coordinates of $\psi^\infty(x)$ and $\psi_n^\infty(x)$ are $\psi(\sigma^i(x))$ and $\psi_n(\sigma^i(x))$ respectively. Thus, for every x:

$$\sup_{x\in\mathcal{C}^{\mathbb{Z}}} \overline{d'}\left(\psi^\infty(x), \psi_m^\infty(x)\right) = \sup_{x\in\mathcal{C}^{\mathbb{Z}}} \sup_{i\in\mathbb{Z}} \frac{1}{2^{|i|}} d'\left(\psi(\sigma^i(x)), \psi_n(\sigma^i(x))\right) \leq \sup_{y\in\mathcal{C}^{\mathbb{Z}}} d'\left(\psi(y), \psi_m(y)\right).$$

We can conclude: the sequence of continuous sliding block codes $(\psi_m^\infty)_{m\in\mathbb{N}}$ converges toward the morphism ψ^∞. □

A.5 Proof of Proposition 1

We denote by $\mathtt{Run}_{\mathcal{G}}$ the set of bi-infinite runs of $\mathcal{G}$: $\mathtt{Run}_{\mathcal{G}} \stackrel{\text{def}}{=} \{(s_i, \alpha_i)_{i\in\mathbb{Z}} \mid s_{i+1} = s_i \triangleright \alpha_i\}$. This set is shift invariant. We show that it is closed. The set $\{(s, \alpha, s') \mid s' = s \triangleright \alpha\}$ is a closed subset of $\mathbb{S} \times \mathbb{A} \times \mathbb{S}$. Indeed, it suffices to remark that for every transition δ the set of tuples $(\boldsymbol{x}, t, \boldsymbol{x}')$ such that $(\delta^-, \boldsymbol{x}) \triangleright (t, \delta) = (\delta^+, \boldsymbol{x}')$ is a polytope defined by constraints involving equality and non-strict inequality only (they have the following form $x' = x + t$, $x' = 0$, $A \leq x + t \leq B$). For a fixed $j \in \mathbb{Z}$ the projection $(s_i, \alpha_i)_{i\in\mathbb{Z}} \mapsto (s_j, \alpha_j, s_{j+1})$ is continuous and hence the set of runs $(s_i, \alpha_i)_{i\in\mathbb{Z}}$ such that $s_{j+1} = s_j \triangleright \alpha_j$ is closed. The set $\mathtt{Run}_{\mathcal{G}}$ is the intersection for $j \in \mathbb{Z}$ of the closed sets described just above, it is thus closed. □

A.6 Proof of Proposition-definition 1

This set is obtained by projecting the set of bi-infinite runs $\mathtt{Run}_{\mathcal{G}}$ on the timed transition components. It is clearly shift-invariant as $\mathtt{Run}_{\mathcal{G}}$. By continuity of the projection, the projected set is compact. □

A.7 Proof of Proposition 3

This proof relies on characterisation of entropies using factors stated in Proposition 5 given in Appendix A.2 above.

The points of $X_{\varepsilon',n}$ are 2ε-separated which proves the first inequality. The second inequality is a straightforward corollary of Lemma 1. To prove the third inequality it suffices to prove that for all $n \in \mathbb{N}$, $X_{\varepsilon,n}$ is an ε-net of X_n. We will adapt a method used in a paper of Henzinger, Manna and Pnueli[17].

The setting of this latter paper considers increasing sequences of dates when events occur instead of delays between events. There is a one-to-one correspondence ϕ between n-uplets of delays and n-uplets of dates defined by $\phi(t_1, ..., t_n) = (t_1, t_1 + t_2, ..., t_1 + t_2 + ... + t_n)$ and with ϕ^{-1} defined by $\phi^{-1}(T_1, ..., T_n) = (T_1, T_2 - T_1, ..., T_n - T_{n-1})$. One can remark that ε-discrete points are also in one to one correspondence by ϕ. The guards along a path give the following inequations on dates $T_k - T_j \in [A, B]$ (this corresponds to a constraint $x \in [A, B]$ checked at the k^{th} transition and with the last reset of x done in the j^{th} transition).

For all real T we denote by $[T]$ the closest multiple of ε to T ($|[T]-T| \leq \frac{\varepsilon}{2}$):

$$[T] = \begin{cases} \varepsilon\lfloor T/\varepsilon \rfloor & \text{if } T \leq \varepsilon\lfloor T/\varepsilon \rfloor + \frac{\varepsilon}{2} \\ \varepsilon(\lfloor T/\varepsilon \rfloor + 1) & \text{otherwise} \end{cases}$$

One can remark that $T_k - T_j \in [A, B]$ implies $[T_k] - [T_j] \in [A, B]$.

Let $(t_1, \delta_1), ..., (t_n, \delta_n) \in X_n$. We denote by $(T_1, ..., T_n) = \phi(t_1, ..., t_n)$ and thus $([T_1], ..., [T_n])$ satisfy the constraints $[T_k] - [T_j] \in [A, B]$. We denote by $(u_1, ..., u_n) = \phi^{-1}([T_1], ..., [T_n])$ then $(u_1, \delta_1), ..., (u_n, \delta_n) \in X_{\varepsilon,n}$. Since $|u_1 - t_1| = |[T_1] - T_1| \leq \frac{\varepsilon}{2}$ then for all $i \in \{2, .., n\}$ we have $|u_i - t_i| = |([T_{i+1}] - [T_i]) - (T_{i+1} - T_i)| \leq |[T_{i+1}] - T_{i+1}| + |[T_i] - T_i| \leq \varepsilon$. Finally every timed word of X_n is at most ε far apart from a timed word of $X_{\varepsilon,n}$. $X_{\varepsilon,n}$ is thus an ε net for X_n which concludes the proof. □

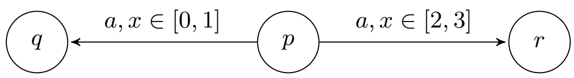

Fig. 3. A right resolving LTG to illustrate the proof of Theorem 4

A.8 Proof of Proposition 4

It is straightforward that $[\mathcal{A}_\varepsilon] \subseteq [\mathcal{A}]_\varepsilon$. To prove the converse inclusion we take $(\texttt{Lab}(\alpha_i))_{i\in\mathbb{Z}} \in [\mathcal{A}]_\varepsilon$ and show that $(\texttt{Lab}(\alpha_i))_{i\in\mathbb{Z}} \in [\mathcal{A}_\varepsilon]$. By definition of $[\mathcal{A}]_\varepsilon$, each α_i is ε-discrete i.e. of the form $\alpha_i = (k_i\varepsilon, \delta_i)$ with $k_i \in \{0, \dots, M/\varepsilon\}$ and $(\alpha_i)_{i\in\mathbb{Z}}$ is obtained by projecting states of an infinite run of $\mathcal{A}$, say $(s_i, \alpha_i)_{i\in\mathbb{Z}}$. We denote by x_i the value of the clock x at the index i of this run. Our objective is to transform the value of x_i for all clocks x and indexes i in such a way that the new values are multiple of ε and the guards are still satisfied. For every clock x, we denote by $\texttt{fr}(x)$ the index of first reset of x (possibly equal to $-\infty$ if the clock is reset infinitely often in the past or $+\infty$ if the clock is never reset).

Since all the delays are multiple of ε then so is x_i for $i \geq \texttt{fr}(x)$. Remark that for all $i < \texttt{fr}(x)$ we have $x_i = x_{\texttt{fr}(x)} - \sum_{i\leq l<\texttt{fr}(x)} k_l\varepsilon$. As all $x_j \in \varepsilon\mathbb{N}$ for $j \in \mathbb{Z}$, it holds that $k_l = 0$ for every l lower than a position $\texttt{fp}(x)$ where it is positive for the first time (here also $\texttt{fp}(x)$ can take values $-\infty$, $+\infty$). We have thus the three possible cases for x_i:

- $x_i = x_{\texttt{fp}(x)}$ if $i \leq \texttt{fp}(x)$;
- $x_i = x_{\texttt{fp}(x)} + \sum_{l=\texttt{fp}(x)}^{i-1} k_l\varepsilon$ if $\texttt{fp}(x) < i \leq \texttt{fr}(x)$;
- x_i is multiple of ε if $i > \texttt{fr}(x)$.

It remains to choose a new value for $x_{\texttt{fp}(x)}$ that is multiple of ε. The guards on the path $(\delta_i)_{i\in Z}$ give inequalities of the form $A_i \leq x_i \leq B_i$. The lower bound for $\varepsilon^{-1}x_{\mathtt{fp}(x)}$ is

$$\sup(\sup_{i\leq \mathtt{fp}(x)} \varepsilon^{-1}A_i, \sup_{\mathtt{fp}(x)<i\leq \mathtt{fr}(x)} \varepsilon^{-1}A_i - \sum_{l=\mathtt{fp}(x)}^{i-1} k_i).$$

This lower bound is an integer since it is a supremum over a set of integers. A symmetric reasoning can be used for the upper bound. An arbitrary choice between the lower and upper bounds gives a new value for $\varepsilon^{-1}x_{\mathtt{fp}(x)}$ which is an integer. This choice does not affect the delays nor the values of the other clocks, it permits to have a new run satisfying the same constraints. One can repeat this operation until all the clocks are ε-discrete in all positions and then we are done. □

A.9 Proof of Theorem 4

We begin by proving the equality of ε-entropies. Remark that $\mathcal{G}$ can be seen as a right-resolving LTG with the labelling function being the identity on Δ. Thus $[\mathcal{G}_\varepsilon]$ and $[\mathcal{A}_\varepsilon]$ are two right-resolving LTG with the same underlying graph. A classical result on symbolics dynamics states that the entropy of a sofic shift of a right-resolving labelled graph is the same as the entropy of the edge shift of the underlying (unlabelled) graph (see [19]). Thus $h(X_{\mathcal{G}_\varepsilon}) = h(X_{\mathcal{A}_\varepsilon})$. We conclude using Corollary 1: $h_\varepsilon([\mathcal{A}]) = h([\mathcal{A}_\varepsilon]) = h([\mathcal{G}_\varepsilon]) = h_\varepsilon([\mathcal{G}])$.

Now we prove the equality $\mathcal{H}([\mathcal{A}]) = \mathcal{H}([\mathcal{G}])$. For each w of length n we have

$$[\mathcal{A}]_n^{|w} = \cup_{\pi\in\mathtt{Lab}^{-1}(w)}[\mathcal{G}]_n^{|\pi} \tag{5}$$

where we recall that $[\mathcal{G}]_n^{|\pi} \overset{\text{def}}{=} \{(t_1,\ldots,t_n) \mid (t_1,\pi_1)\ldots(t_n,\pi_n) \in [\mathcal{G}]_n\}$ and $[\mathcal{A}]_n^{|w} \overset{\text{def}}{=} \{(t_1,\ldots,t_n) \mid (t_1,w_1)\ldots(t_n,w_n) \in [\mathcal{A}]_n\}$. Then for every $n \in \mathbb{N}$:

$$\mathtt{Vol}([\mathcal{A}]_n) = \sum_{w\in\Sigma^n} \mathtt{Vol}([\mathcal{A}]_n^{|w}) \leq \sum_{w\in\Sigma^n} \sum_{\pi\in\mathtt{Lab}^{-1}(w)} \mathtt{Vol}([\mathcal{G}]_n^{|\pi}) = \mathtt{Vol}([\mathcal{G}]_n).$$

This implies $\mathcal{H}([\mathcal{A}]) \leq \mathcal{H}([\mathcal{G}])$.

For the converse inequality, everything would be very simple if the union in (5) were disjoint. This is not the case due to freedom on the initial state. Indeed, let us consider for instance the right resolving LTG depicted in Figure 3. The timed transition $(0.5, a)$ taken from $(p, 0.2)$ leads to $(q, 0.7)$ while taken from $(p, 1.8)$ leads to $(r, 2.3)$. However the polytopes $P_{\pi_{(2)}}(s)$, $P_{\pi'_{(2)}}(s)$ of delays that can be read along two distinct paths $\pi_{(2)} \neq \pi'_{(2)}$ starting from a state s are disjoint due to right-resolvness. Then, if all clocks are reset during a path $\pi_{(1)}$ of length l (for $l > 0$), for every two distinct paths $\pi_{(2)}, \pi'_{(2)}$ of length $n - l$ (for $n > l$) we have $[\mathcal{G}]_n^{|\pi_{(1)}\pi_{(2)}} \cap [\mathcal{G}]_n^{|\pi_{(1)}\pi'_{(2)}} = \emptyset$. We divide into two groups the set of paths of a given length l (l is a parameter that we will tune later):

- the set $R(l)$ of paths which reset all its clocks;

– the set of other paths. For paths in this latter set, it holds that $\texttt{Vol}([\mathcal{G}]_n^{|\pi}) \leq \frac{M^n}{l!}$.

It holds that

$$\begin{aligned}\texttt{Vol}([\mathcal{G}]_n) &= \sum_{\pi \in \Delta^n} \texttt{Vol}([\mathcal{G}]_n^{|\pi}) \\ &= \sum_{\pi_{(1)} \in R(l)} \sum_{\pi_{(2)} \in \Delta^{n-l}} \texttt{Vol}([\mathcal{G}]_n^{|\pi_{(1)}\pi_{(2)}}) + \sum_{\pi_{(1)} \notin R(l)} \sum_{\pi_{(2)} \in \Delta^{n-l}} \texttt{Vol}([\mathcal{G}]_n^{|\pi_{(1)}\pi_{(2)}}).\end{aligned}$$

We will denote by S_1 and S_2 the two sums above. S_2 is upper bounded by $\frac{(|\Delta|M)^n}{l!}$. For each $w \in \Sigma^n$ we have

$$[\mathcal{A}]_n^{|w} = \bigcup_{\pi_{(1)} \in R(l)} \biguplus_{\substack{\pi_{(2)} \in \Delta^{n-l} \\ \pi_{(1)}\pi_{(2)} \in \texttt{Lab}^{-1}(w)}} [\mathcal{G}]_n^{|\pi_{(1)}\pi_{(2)}}$$

and then

$$\begin{aligned}\texttt{Vol}([\mathcal{A}]_n^{|w}) &\geq \max_{\pi_{(1)} \in R(l)} \sum_{\substack{\pi_{(2)} \in \Delta^{n-l} \\ \pi_{(1)}\pi_{(2)} \in \texttt{Lab}^{-1}(w)}} \texttt{Vol}([\mathcal{G}]_n^{|\pi_{(1)}\pi_{(2)}}) \\ &\geq \frac{1}{|\Delta|^l} \sum_{\pi_{(1)} \in R(l)} \sum_{\substack{\pi_{(2)} \in \Delta^{n-l} \\ \pi_{(1)}\pi_{(2)} \in \texttt{Lab}^{-1}(w)}} \texttt{Vol}([\mathcal{G}]_n^{|\pi_{(1)}\pi_{(2)}}).\end{aligned}$$

We sum over all w and deduce that $\texttt{Vol}([\mathcal{A}]_n) \geq S_1/|\Delta|^l$. Remark that $\mathcal{H}([\mathcal{A}]) \geq \mathcal{H}([\mathcal{G}]) - \log_2(|\Delta|)$ since

$$\texttt{Vol}([\mathcal{A}]_n) = \sum_{w \in \Sigma^n} \texttt{Vol}([\mathcal{A}]_n^{|w}) \geq \max_{\pi \in \Delta^*} \texttt{Vol}([\mathcal{G}]_n^{|\pi}) \geq \frac{\texttt{Vol}([\mathcal{G}]_n)}{|\Delta|^n}.$$

Hence $\mathcal{H}([\mathcal{A}]) = -\infty$ iff $\mathcal{H}([\mathcal{G}]) = -\infty$. We suppose now that $\mathcal{H}([\mathcal{A}]) > -\infty$. In particular $\texttt{Vol}([\mathcal{A}]_n)$ behaves like an exponent in the following sense: for every $a < 2^{\mathcal{H}([\mathcal{A}])} < b$ it holds that $b^n >> \texttt{Vol}([\mathcal{A}]_n) >> a^n$.

Recap that $\texttt{Vol}([\mathcal{G}]_n) = S_1 + S_2$, $S_2 \leq \frac{|\Delta|^n}{l!}$ and $S_1 \leq \texttt{Vol}([\mathcal{A}]_n)|\Delta|^l$, hence

$$\texttt{Vol}([\mathcal{G}]_n) \leq \texttt{Vol}([\mathcal{A}]_n) \left(|\Delta|^l + \frac{|\Delta|^n}{\texttt{Vol}([\mathcal{A}]_n) l!} \right). \qquad (6)$$

We choose l such that $l << n << \log_2(l!)$, for instance l such that $n = \lfloor l \log_2(\log_2(l)) \rfloor$. With such an l the quantity $\frac{1}{n} \log_2 \left(|\Delta|^l + \frac{(|\Delta|M)^n}{\texttt{Vol}([\mathcal{A}]_n) l!} \right)$ tends to 0 and then taking $\lim_{n \to \infty} \frac{1}{n} \log_2(.)$ in (6) yields the expected inequality: $\mathcal{H}([\mathcal{G}]) \leq \mathcal{H}([\mathcal{A}])$. □

A.10 Proof of Lemma 2

There is $\binom{n+M/\varepsilon}{n}$ possibilities to choose n indices $i_1 < \ldots < i_n$ among $\{1, \ldots, n+M/\varepsilon\}$. For $j = 1..n$ define $u_j = (i_j - j)\varepsilon$ and get $0 \leq u_1 \leq \cdots \leq u_n \leq M$. It remains to remark that the mapping $(i_1, \ldots, i_n) \mapsto (t_1, \ldots, t_n)$ is a bijection (one can check that the following function is an inverse for it: $u_j \mapsto u_j/\varepsilon + j$ for $j = 1..n$). □

A.11 Proof of Theorem 5

Examples with metric mean dimension 0 or 1 have already been treated (X^{I} and X^{II}). For every $k, l \in \mathbb{N}$ such that $0 < k < l$, we describe a cyclic fleshy LTG with $2l$ transitions and metric mean dimension $k/l \in \mathbb{Q} \cap (0, 1)$. The transitions are $q_i \xrightarrow{a, x\in[i,i+1]} q_{i+1}$ for $i = 0..2k-1$; $q_i \xrightarrow{b, y\geq 2k+1, x\in[2k,2k+1]} q_{i+1}$ for $i = 2k..2l-3$ (there is no transition labelled by b if $2l-3 < 2k$ that is if and only if $l = k+1$); $q_{2l-2} \xrightarrow{c, y\geq 2k+1, x\in[2k,2k+1], y:=0} q_{2l-1}$ and $q_{2l-1} \xrightarrow{d, x\in[2k,2k+1], x:=0} q_0$ (remark that $2l-2 \geq 2k$ so that the index are well defined).

We consider wlog. only runs such that the location in index 0 is q_0 (indeed shifting runs does not change mean properties such as the metric mean dimension). Let $(s_n, \alpha_n)_{n\in\mathbb{Z}}$ be such a run with the following notation $s_n = (q_n, (x_n, y_n))$ and $\alpha_n = (t_n, \delta_n)$.

Both clocks are not reset between locations q_0 and q_{2k+1} then $y_{2k+1} - y_0 = x_{2k+1} - x_0 = x_{2k+1}$ and thus $y_0 = y_{2k+1} - x_{2k+1} \geq 2k+1 - x_{2k+1}$ using the guard on the transition from q_{2k} to q_{2k+1} labelled by b.

Since x is not reset between indices $2k+1$ and $2l-1$ it holds that $2k+1 - x_{2l-1} \leq 2k+1 - x_{2l-2} \leq \ldots \leq 2k+1 - x_{2k+1}$.

Remark that $y_{2l} = t_{2l-1}$ (the clock y is null in q_{2l-1}) and $x_{2l-1} + t_{2l-1} \leq 2k+1$ (by the guard of transition labelled by d), hence $y_{2l} \leq 2k+1 - x_{2l-1}$.

As the timed graph is cyclic the discussion above holds as well when the indices are shifted by a multiple of its length $2l$.

We denote by $z_{2lm} \stackrel{\text{def}}{=} y_{2lm} = t_{2lm-1}$ and by $z_{2lm+i} \stackrel{\text{def}}{=} 2k+1 - x_{2lm+i}$ for all $m \in \mathbb{N}$ and $i = 2k+1..2l-1$. Remark that $x_{2lm+i} = \sum_{i=2lm}^{2lm+i-1} t_i$ since the last reset of x before $2lm+i$ is just before entering q_{2lm}. Remark that the change of coordinate from t_j to z_{j+1} for $j \neq 0..2k-1 \pmod{2l}$ and that leave the other t_j unchanged is bijective between ε-discrete points. The coordinates z_j lie in the following bi-infinite simplex: $0 \leq \ldots z_{2lm} \leq z_{2lm+2k+1} \leq \ldots \leq z_{2l(m+1)} \leq \ldots \leq 1$: the t_j for $j = 0..2k-1 \pmod{2l}$ can be chosen independently and hence there is $1/\varepsilon$ choices per such delays. Using Lemma 2 we have that the number of point in the polytope corresponding to m cycles starting from q_0 is $\binom{2lm+1/\varepsilon}{2lm}(1/\varepsilon)^{2km}$. Then we compute $(1/2lm)\log_2[\binom{nm+1/\varepsilon}{nm}(1/\varepsilon)^{km}] = (1/2lm)\log_2[\binom{2lm+1/\varepsilon}{2lm}] + (2k/2l)\log_2(1/\varepsilon)$. Now we let $m \to +\infty$ and the first vanishes. Dividing the remaining term by $\log_2(1/\varepsilon)$ (and letting $\varepsilon \to 0$) yields the expected metric mean dimension k/l. □